Assignment 5

February 23, 2018

1. Suppose $X_,..., X_n \overset{iid}{\sim} f_\theta(x)$ and let β denotes the power of the most powerful level α test for testing $H_0 : \theta = 0$ vs. $H_1 : \theta = 1$. Show that $\alpha \le \beta$. Use the same logic to conclude that if ϕ is a UMP level α test then ϕ is unbiased.

2. Suppose X has a distribution P_θ for some $\theta \in \Omega$, and the null hypothesis $H_0 : \theta \in \Omega_H$. Assume that we have a test whose rejection region satisfy $\mathcal{R}_\alpha \subset \mathcal{R}_{\alpha'}$ for $\alpha < \alpha'$.

   (a) Show that for all $\theta \in \Omega_H$, $P_\theta(p(X) \le u) \le u$ for all $0 \le u \le 1$.

   (b) If, for $\theta \in \Omega_H$, $P_\theta(X \in \mathcal{R}_\alpha) = \alpha$ for all $0 \le \alpha \le 1$, then $P_\theta(p(\boldsymbol{X}) \le u) = u$ for all $0 \le u \le 1$.

3. Let $X_1, ..., X_n \overset{iid}{\sim} Pois(\lambda)$. Find the UMP level $\alpha = 1 - e^{-n\lambda_0}(1 + n\lambda_0)$ test for $H_0 : \lambda \le \lambda_0$ vs. $H_1 : \lambda > \lambda_0$.

4. Let $X_1, ..., X_n \overset{iid}{\sim} U(0,\theta)$. Find the UMP level α test for testing $H_0 : \theta = \theta_0$ vs. $H_1 : \theta \neq \theta_0$.

5. Let X_1, X_2 be iid $U(\theta, \theta+1)$. For testing $H_0 : \theta = 0$ vs. $H_1 : \theta > 0$, we have two competing tests,

$$\phi_1(X_1) = 1 \text{ if } X_1 > .95; \ = 0 \text{ o.w.}$$

$$\phi_2(X_1, X_2) = 1 \text{ if } X_1 + X_2 > C; \ = 0 \text{ o.w.}$$

   (a) Find the value of C so that ϕ_2 has the same size as ϕ_1.

   (b) Calculate the power function of each test.

   (c) Show how to get a test that has the same size but is more powerful than ϕ_2.

6. Show that for a random sample $X_1, ..., X_n \overset{iid}{\sim} N(0, \sigma^2)$. Find most powerful level α test of $H_0 : \sigma = \sigma_0$ vs. $H_1 : \sigma = \sigma_1$, where $\sigma_0 < \sigma_1$.

7. Suppose $X_1, ..., X_n$ are iid with a $Beta(\mu, 1)$ and $Y_1, ..., Y_m$ are iid with a $Beta(\theta, 1)$. Also assume that Xs are independent of Ys.

   (a) Find an LRT of $H_0 : \theta = \mu$ vs. $H_1 : \theta \neq \mu$.

   (b) Show that the test above can be based on the statistic $T = \frac{\sum_{i=1}^n \log(X_i)}{\sum_{i=1}^n \log(X_i) + \sum_{i=1}^m \log(Y_i)}$.

8. Suppose that we have two independent random samples: $X_1, ..., X_n$ are $Exp(\theta)$ and $Y_1, ..., Y_m$ are $Exp(\mu)$.

(a) Find the LRT of $H_0 : \theta = \mu$ vs. $H_1 : \theta \neq \mu$.

(b) Show that the test above can be based on the statistic $T = \frac{\sum_{i=1}^n X_i}{\sum_{i=1}^n X_i + \sum_{i=1}^m Y_i}$.

(c) Find the distribution of T when H_0 is true.

9. Let $X_1, ..., X_n$ be a random sample from a $N(\mu_X, \sigma_X^2)$, and let $Y_1, ..., Y_m$ be an independent random sample from a $N(\mu_Y, \sigma_Y^2)$. We are interested in testing $H_0 : \mu_X = \mu_Y$ vs. $H_1 : \mu_X \neq \mu_Y$, with the assumption that $\sigma_X^2 = \sigma_Y^2$. Derive the LRT test for this hypothesis. Provide an LRT test of size α.

10. Let $X_1, ..., X_n$ be a random sample from a $N(\theta, \sigma^2)$ population. Consider testing $H_0 : \theta \leq \theta_0$ vs. $H_1 : \theta > \theta_0$. Let $\bar{X}_m$ denote the sample mean of the first m observations, $X_1, ..., X_m$, for $m = 1, ..., n$. The test that rejects H_0 when $\bar{X}_m > \theta_0 + z_\alpha \sqrt{\sigma^2/m}$ is an unbiased size α test.

11. In each of the following situations, calculate the p value of the observed data.

(a) For testing $H_0 : \theta \leq 0.5$ vs. $H_1 : \theta > 0.5$, 7 successes are observed out of 10 Bernoulli trials.

(b) For testing $H_0 : \lambda \leq 1$ vs. $H_1 : \lambda > 1$, $X = 3$ is observed, where $X \sim Pois(\lambda)$.

12. Let X be one observation from a Cauchy(θ) distribution.

(a) Show that the family does not have an MLR.

(b) Show that the test

$$\phi(x) = \begin{cases} 1 \text{ if } 1 < X < 3 \\ 0 \text{ o.w.} \end{cases}$$

is most powerful of its size for testing $H_0 : \theta = 0$ vs. $H_1 : \theta = 1$. Calculate the Type I and Type II error probabilities.

(c) Is it a UMP test for testing $H_0 : \theta = 0$ vs. $H_1 : \theta > 0$? Prove or disprove.