Assignment 4

February 2, 2018

1. Using change of variable theorem calculate the value of \( \int_0^1 \int_0^{1-x} \sqrt{x+y(y-2x)^2} dy \, dx \).

2. \( \mathbf{X} = (X_1, ..., X_k) \sim \text{Dirichlet}(\alpha_1, ..., \alpha_k) \), \( X_k = 1 - \sum_{i=1}^{k-1} X_i \), if the joint p.d.f has the form

\[
 f_{\alpha_1, ..., \alpha_k}(x_1, ..., x_{k-1}) = \frac{\Gamma(\sum_{i=1}^{k} \alpha_i)}{\prod_{i=1}^{k} \Gamma(\alpha_i)} \left[ \prod_{i=1}^{k-1} x_i^{\alpha_i-1} \right] (1 - \sum_{i=1}^{k-1} x_i)_{\alpha_k-1}, \quad x_1, ..., x_{k-1} > 0, \quad \sum_{i=1}^{k-1} x_i < 1.
\]

Prove that if \( Y_i \sim \text{Gamma}(\alpha_i, \theta) \), \( V = \sum_{i=1}^{k} Y_i \), then \( (Y_1 V^{-1}, ..., Y_k V^{-1}) \sim \text{Dirichlet}(\alpha_1, ..., \alpha_k) \).

3. Using the Central Limit theorem show that \( \sum_{i=0}^{n} \exp\left(-n\frac{x^i}{i!}\right) \rightarrow \frac{1}{2} \).

4. Use the result that if \( X_n \rightarrow c \) in probability then \( g(X_n) \rightarrow g(c) \) in probability where \( g \) is continuous in \( c \). Let \( \{Z_n\}_{n \geq 1}, \{Y_n\}_{n \geq 1} \) be two sequences such that \( E[Z_n] = E[Y_n] = 0 \) and \( E[Z_n^2] = E[Y_n^2] = \sigma^2 \). Prove using the above result, Central Limit theorem, Weak Law of Large number and Slutsky’s theorem that

\[
 \frac{Z_1 Y_1 + \cdots + Z_n Y_n}{Z_1^2 + \cdots + Z_n^2} \rightarrow N(0,1), \quad \text{in distribution.}
\]

5. Let \( X, ..., X_n \) be a random sample with cumulative density function (c.d.f) of \( X \) is \( F(x) \), i.e. \( P(X \leq x) = F(x) \). Prove that \( P(X_{(j)} \leq x) = \sum_{r=j}^{n} \binom{n}{r} (F(x))^r (1 - F(x))^{n-r} \).

6. Using Delta theorem prove that
• If \(X_1, ..., X_n \overset{iid}{\sim} Ber(p), 0 < p < 1\) then \(\sqrt{n}[-1(\sqrt{X_n}) - \sin^{-1}(\sqrt{p})] \to N(0, 1/4)\).

• If \(X_1, ..., X_n \overset{iid}{\sim} N(\mu, \sigma^2)\), then \(\sqrt{n}[-\log(S_n) - \log(\sigma^2)] \to N(0, 2)\), where \(S_n = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2\).

Notice that in all these cases the asymptotic variance is free of any parameter. Such a transformation is known as variance stabilizing transformation of the statistic.

7. If \(X\) follows a multi-parameter exponential family density and \(g\) is any differentiable function s.t. \(E_\theta|g'(X)| < \infty\) then

\[
E\left[\left\{ \frac{h'(X)}{h(X)} + \sum_{i=1}^{k} w_i(\theta)T_i'(X) \right\} g(X) \right] = -E[g'(X)],
\]

provided the support of \(X\) is \((-\infty, \infty)\).

8. Let \(X_1, ..., X_n\) be iid geometric distribution with \(P_\theta(X = x) = \theta(1-\theta)^{x-1}, x = 1, 2, ..., 0 < \theta < 1\). Show that \(\sum_{i=1}^{n} X_i\) is sufficient and complete for \(\theta\).

9. Let \((X_1, X_2, X_3, X_4) \sim Multinomial\) with cell probabilities \(\frac{1}{2} + \frac{\theta}{4}, \frac{1}{4}(1-\theta), \frac{1}{4}(1-\theta), \frac{\theta}{4}\).

Find a sufficient and minimal sufficient statistic for \(\theta\).

10. Let \(N\) be a random variable taking values 1, 2, ... with known probabilities \(p_1, ...\) where \(\sum_i p_i = 1\). Having observed \(N = n\), perform \(n\) Bernoulli trials with success probability \(\theta\), getting \(X\) successes.

- Prove that the pair \((X, N)\) is minimal sufficient and \(N\) is ancillary for \(\theta\).
- Prove that the estimator \(\frac{X}{N}\) is unbiased for \(\theta\) and has variance \(\theta(1-\theta)E[\frac{1}{N}]\).

11. \(X_1, ..., X_{2m+1} \sim N(\mu, \sigma^2), \sigma^2\) known. Prove that \(Var(X_{(m+1)}) \geq Var(\bar{X}_{2m+1})\).

12. Let \(X_1, ..., X_n \overset{iid}{\sim} f(\frac{z-\mu}{\sigma})\). Let \(T_1(X_1, ..., X_n)\) and \(T_2(X_1, ..., X_n)\) be two statistics that both satisfy \(T_1(ax_1 + b, ..., ax_n + b) = aT_1(x_1, ..., x_n)\), for all values of \(x_1, ..., x_n\) and \(b\) and for any \(a > 0\).
• Show that $T_1(X)/T_2(X)$ is an ancillary statistic.

• Let $R$ be the sample range $R = X_{(n)} - X_{(1)}$ and $S = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2$. Then $R/S$ is an ancillary statistic.

13. Let $X$ takes on values $-1, 0, 1, 2, ...$ with probabilities

\[ P(X = -1) = p, \quad P(X = k) = (1 - p)^2 p^k, \quad k = 0, 1, ... \]

where $0 < p < 1$.

• Prove that $U(X)$ is an unbiased estimator of 0 if and only if $U(k) = -kU(-1)$ for all $k = 0, 1, ...$.

• Let

\[ \delta_0(X) = \begin{cases} 
1 & \text{if } X = -1 \\
0 & \text{otherwise} 
\end{cases} \]

$\delta_0$ is an unbiased estimator of 0. If $U(-1) \neq 0$, prove that $\delta_0$ cannot be the UMVUE.

14. $X_1, ..., X_n \sim \text{Bernoulli}(p)$. Provide the UMVUE of $p(1 - p)$.

15. $X_1, ..., X_n \sim N(\mu, 1)$. Provide UMVUE of $P(X_1 \leq u) = \Phi(u - \mu)$ for some $u$.

16. $X_1, ..., X_n \sim f_{\theta_1, \theta_2}(x)$ where

\[ f_{\theta_1, \theta_2}(x) = \frac{1}{\theta_2} \exp(-(x - \theta_1)/\theta_2), \quad x > \theta_1. \]

• Show that $X_{(1)}$ and $\sum_{i=1}^{n} [X_i - X_{(1)}]$ are jointly sufficient and complete.

• Show that $X_{(1)}$ and $\sum_{i=1}^{n} [X_i - X_{(1)}]$ are independent.

• Find UMVUE for $\theta_1$ and $\theta_2$. 

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17. \( X_1, ..., X_n \overset{iid}{\sim} U(0, \theta), Y_1, ..., Y_m \overset{iid}{\sim} U(0, \theta_1) \) and all \( X \) and \( Y \) are independent. Find UMVUE for \( \theta/\theta' \).

18. \( X_1, ..., X_n \overset{iid}{\sim} \text{Pois}(\lambda) \). Prove that \( E[S^2|\bar{X}] = \bar{X} \), hence \( \text{var}(S^2) < \text{var}(\bar{X}) \).

19. Suppose \( X \) and \( Y \) are independently distributed with densities \( f_\theta \) and \( q_\theta \) respectively. If \( I_1(\theta), I_2(\theta) \) and \( I(\theta) \) are the information of \( X, Y \) and \( (X,Y) \) respectively, prove that \( I(\theta) = I_1(\theta) + I_2(\theta) \).

20. For each of the following distributions, let \( X_1, ..., X_n \) be a random sample. Is there a function of \( \theta \), say \( g(\theta) \), for which there exists an unbiased estimator whose variance attains the Cramer-Rao lower bound? If so, find it.

   a. \( f_\theta(x) = \theta x^{\theta-1}, 0 < x < 1, \theta > 0. \)

   b. \( f_\theta(x) = \frac{\log(\theta)}{\theta-1} x^x, 0 < x < 1, \theta > 1. \)

21. Let \( X_1, ..., X_n \) be a random sample from a population with p.m.f \( P_\theta(X = x) = \theta^x(1-\theta)^{1-x}, x = 0, 1, 0 \leq \theta \leq 1/2. \)

   a. Find the method of moment estimator and MLE of \( \theta \).

   b. Find the mean squared errors for each of them.

   c. Which one do you prefer based on mean squared errors?

22. Let \( X_1, ..., X_{2m+1} \) follows iid \( f_\theta(x) = \frac{1}{2} \exp(-|x-\theta|), -\infty < x < \infty. \) Find the MLE of \( \theta. \)

23. The LINEX loss is given by

\[
L(\theta, a) = e^{c(a-\theta)} - c(a-\theta) - 1,
\]

where \( c \) is a positive constant. Show that the Bayes estimator of \( \theta \) under this loss function, using a prior \( \pi \) is \( \delta^\pi(X) = \frac{-1}{c} \log[E(e^{-c|X|})]. \) Evaluate this estimator when
$X_1, \ldots, X_n \sim N(\theta, 1)$ and $\pi(\theta) = 1$. 