Assignment 2

January 19, 2018

1. If $X_1, ..., X_n \overset{iid}{\sim} Beta(1, \beta)$, find a value of ν so that $n^\nu(1 - X_{(n)})$ converges in distribution.

2. Let $X_1, ..., X_n \overset{iid}{\sim} Bernoulli(p)$ and $Y_n = \frac{1}{n}\sum_{i=1}^n X_i$.

   - Show that $\sqrt{n}(Y_n - p) \to N(0, p(1-p))$ in distribution.
   - Show that for $p \neq 0.5$, the estimate of variance $Y_n(1 - Y_n)$ satisfies $\sqrt{n}[Y_n(1 - Y_n) - p(1-p)] \to N(0, p(1-p)(1-2p)^2)$ in distribution.
   - Show that for $p = 0.5$, $n[Y_n(1 - Y_n) - \frac{1}{4}] \to -\frac{1}{4}\chi_1^2$ in distribution.

3. Let $X_1, ..., X_n$ be independent random random variables with densities

$$f_{X_i,\theta}(x) = \begin{cases} e^{i\theta - x} & \text{if } x \geq i\theta \\ 0 & \text{if } x < i\theta \end{cases}$$

   Prove that $T = min_i(X_i/i)$ is a sufficient statistic for θ.

4. Let $X_1, ..., X_n$ be a random sample from the pdf

$$f_{\mu,\sigma}(x) = \begin{cases} \frac{1}{\sigma}e^{-(x-\mu)/\sigma} & \text{if } \mu < x < \infty,\ 0 < \sigma < \infty \\ 0 & \text{o.w.} \end{cases}$$

   Find a two-dimensional sufficient statistic for (μ, σ).

5. $X_1, ..., X_n$ be independent random variables with pdfs

$$f_{X_i,\theta}(x) = \begin{cases} \frac{1}{2i\theta} \text{ if } -i(\theta-1) < x < i(\theta+1) \\ 0 \text{ o.w.} \end{cases}$$

where $\theta > 0$, find a two-dimensional sufficient statistic for θ.

6. Let $X_1, ..., X_n$ be a random sample from a $Gamma(\alpha, \beta)$ distribution. Find a two dimensional sufficient statistics for (α, β).

7. For each of the following distributions let $X_1, ..., X_n$ be a random sample. Find a minimal sufficient statistic for θ.

   - $f_\theta(x) = \frac{1}{\sqrt{2\pi}} e^{-(x-\theta)^2/2}$, $-\infty < x < \infty$, $-\infty < \theta < \infty$.
   - $f_\theta(x) = e^{-(x-\theta)}$, $\theta < x < \infty$, $-\infty < \theta < \infty$.
   - $f_\theta(x) = \frac{1}{\pi[1+(x-\theta)^2]}$, $-\infty < x < \infty$, $-\infty < \theta < \infty$.

8. Suppose X_1, X_2 are iid observations from the pdf $f_\alpha(x) = \alpha x^{\alpha-1} e^{-x^\alpha}$, $x > 0$, $\alpha > 0$. Show that $\log(X_1)/\log(X_2)$ is an ancillary statistic.

9. Let $X_1, ..., X_n$ be a random sample from a location family. Prove that $M - \bar{X}$ is an ancillary statistic where M is the sample median and $\bar{X}$ is the sample mean.