Assignment 1

January 12, 2018

1. Let $X_1, ..., X_n$ be iid random variables with continuous cdf F_X, and suppose $E(X_i) = \mu$. Define the random variables $Y_1, ..., Y_n$ by $Y_i = 1$ if $X_i > \mu$; $Y_i = 0$ o.w. Find the distribution of $\sum_{i=1}^{n} Y_i$.

2. If $X_i \sim Binomial(n_i, p)$ and X_i are independent for $i = 1, ..., k$, find the distribution of $\sum_{i=1}^{k} X_i$.

3. A generalization of iid random variables is *exchangeable* random variables, an idea due to deFinetti (1972). The random variables $X_1, ..., X_n$ are *exchangeable* if any permutation of any subset of them of size k ($k \leq n$) has the same distribution. In the exercise we will see an example of random variables that are exchangeable but not iid. Let $X_i|p \overset{iid}{\sim} Bernoulli(p)$, $i = 1, ..., n$ and $p \sim U(0, 1)$.

   - Show that the marginal distribution of any k of the Xs is the same as
   $$P(X_1 = x_1, ..., X_k = x_k) = \int_0^1 p^t (1-p)^{k-t} dp,$$
   where $t = \sum_{i=1}^{k} x_i$. Argue that the Xs are exchangeable.
   - Show that marginally $P(X_1 = x_1, ..., X_n = x_n) \neq P(X_1 = x_1) \cdots P(X_n = x_n)$. Hence Xs are not iid.

4. $X \sim Bibomial(n, p)$. Find MGF of X. Also find the MGF of X when $X \sim Pois(\lambda)$.

5. Let $X, Y \overset{iid}{\sim} N(0,1)$ random variables, and define $Z = min(X, Y)$. Prove that $Z^2 \sim \chi_1^2$.

6. Suppose $X_1, ..., X_n \overset{iid}{\sim} N(\mu, \sigma^2)$. Find a function of S^2, the sample variance, say $g(S^2)$ that satisfies $E(g(S^2)) = \sigma$. Try $g(S^2) = c\sqrt{S^2}$, where c is a constant. Try to find c.

7. $X_1, ..., X_n$ be a random sample from a population with pdf

$$f_X(x) = \begin{cases} 1/\theta, \text{ if } 0 < x < \theta \\ 0 \text{ otherwise} \end{cases}$$

Let $X_{(1)} < \cdots < X_{(n)}$ be the order statistics. Show that $X_{(1)}/X_{(n)}$ and $X_{(n)}$ are independent random variables.

8. As a generalization of the previous exercise, let $X_1, ..., X_n$ be iid with pdf

$$f_X(x) = \begin{cases} \frac{a}{\theta^a} x^{a-1}, \text{ if } 0 < x < \theta \\ 0 \text{ otherwise} \end{cases}$$

Let $X_{(1)} < \cdots < X_{(n)}$ be the order statistics. Show that $X_{(1)}/X_{(2)}$, $X_{(2)}/X_{(3)}$,..., $X_{(n-1)}/X_{(n)}$ and $X_{(n)}$ are mutually independent random variables. Find the distribution of them.