

Goal:

- ① Build a model
- ② Estimate the model parameters
- ③ Evaluate whether one method of estimation is better than the other.
- ④ Hypothesis testing
- ⑤ Uncertainty characterisation/Interval estimation
chapters 5-10, Casella & Berger.

Random sample

x_1, \dots, x_n ~~are~~ together known to be a random sample if they are identically distributed and mutually independent.

Suppose each x_i has a density $f(x|\theta)$, then we mathematically write

$$x_1, \dots, x_n \stackrel{i.i.d}{\sim} f(x|\theta)$$

where $x_1, \dots, x_n \stackrel{i.i.d}{\sim} f(x|\theta)$, the joint density of x_1, \dots, x_n is given by

$$f(x_1, \dots, x_n | \theta) = \prod_{i=1}^n f(x_i | \theta)$$

Ex: $x_1, \dots, x_n \stackrel{iid}{\sim} \text{Exp}(\lambda)$

then,

$$P(x_1 \leq a, \dots, x_n \leq a) = \prod_{i=1}^n P(x_i \leq a)$$

$$= \underbrace{P(x_1 \leq a)}_{P(x_1 \leq a) = 1 - e^{-\lambda a}} \cdots P(x_n \leq a)$$

$$P(x_1 \leq a) = 1 - e^{-\lambda a}, P(x_2 \leq a) = 1 - e^{-\lambda a}, \dots, P(x_n \leq a) = 1 - e^{-\lambda a}$$

$$= [1 - e^{-\lambda a}]^n$$

Result: If x_1, \dots, x_n is a random sample,
then $g(x_1), \dots, g(x_n)$ is also a random sample,
for any function g .

Result: $x_1, \dots, x_n \stackrel{iid}{\sim} \text{Pois}(\lambda)$

~~P~~ Goal: Find $P(x_1 + \dots + x_n = a)$ where a
is ~~a real~~ a positive integer.

$n=2$,

$$P(x_1 + x_2 = a) = P\left(\bigcup_{l=0}^a \{x_1 = l, x_2 = a-l\}\right)$$

$$= \sum_{l=0}^a P(x_1 = l, x_2 = a-l)$$

$$= \sum_{l=0}^a P(x_1 = l) P(x_2 = a-l) \quad \begin{bmatrix} \text{since } x_1, x_2 \text{ are part of} \\ \text{a random sample} \end{bmatrix}$$

$$= \sum_{l=0}^a \frac{e^{-\lambda} \lambda^l}{l!} \frac{e^{-\lambda} \lambda^{a-l}}{(a-l)!}$$

$$= e^{-2\lambda} \sum_{l=0}^a \frac{\lambda^l \lambda^{a-l}}{l! (a-l)!}$$

$$= e^{-2\lambda} \lambda^a \sum_{l=0}^a \frac{1}{l! (a-l)!}$$

$$= e^{-2\lambda} \frac{\lambda^a}{a!} \sum_{l=0}^a \frac{a!}{l! (a-l)!}$$

$$= e^{-2\lambda} \frac{\lambda^a}{a!} \sum_{l=0}^a \binom{a}{l} \quad \left[(1+1)^a = \sum_{l=0}^a \binom{a}{l} \right]$$

$$= e^{-2\lambda} \frac{\lambda^a}{a!} 2^a = e^{-2\lambda} \frac{(2\lambda)^a}{a!}$$

$\bullet P(x_1 + x_2 = a) = e^{-2\lambda} \frac{(2\lambda)^a}{a!}$ for any positive integer a

$$\Rightarrow x_1 + x_2 \sim \text{Pois}(2\lambda)$$

$$x_1 + x_2 + x_3 \sim \text{Pois}(3\lambda) \dots, x_1 + \dots + x_n \sim \text{Pois}(n\lambda)$$

Some Important Facts:

$$E[x^k] = \int x^k f(x|\theta) dx, \text{ for any } k$$

$$E[x_i x_j] = \iint x_i x_j f(x_i|\theta) f(x_j|\theta) dx_i dx_j$$

$$\text{Cov}(x_i, x_j) = E[x_i x_j] - E[x_i] E[x_j]$$

where x_i & x_j are part of a random sample

$$\text{then } \text{Cov}(x_i, x_j) = 0$$

Ans

③

When x_i & x_j are independent $\text{cov}(x_i, x_j) = 0$
 The reverse is not true in general except for
 the normal distribution.

Moment Generating Function

Goal: Calculate $E[x^k]$ for any k when k is
 an integer. Moment generating function for a
 random variable x is defined as

$$E[e^{tx}] = \int e^{tx} f(x|\theta) dx = M_X(t)$$

here $f(x|\theta)$ is the density of x . This function
 $M_X(t)$ is a function of t .

Suppose $M_X(t)$ exists in the nbd. of 0.

$$\textcircled{1} \quad \frac{d}{dt} M_X(t) = \frac{d}{dt} E[e^{tx}] = E\left[\frac{d}{dt} e^{tx}\right] \\ = E[X e^{tx}]$$

$$\frac{d}{dt} M_X(t) \Big|_{t=0} = E[X e^{tx}] \Big|_{t=0} = E[X]$$

$$\frac{d^2}{dt^2} M_X(t) = E\left[\frac{d}{dt} X e^{tx}\right] = E[X^2 e^{tx}]$$

$$\Rightarrow \frac{d^2}{dt^2} M_X(t) \Big|_{t=0} = E[X^2]$$

$$\vdots \\ \frac{d^k}{dt^k} M_X(t) \Big|_{t=0} = E[X^k]$$

Examples: $X \sim N(0, 1)$, $E[X] = 0$, $E[X^{2k+1}] = 0$
for any integer k .

$$\begin{aligned}
 M_X(t) &= E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx \\
 &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{tx - \frac{x^2}{2}\right\} dx \\
 &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}[x^2 - 2tx + t^2 - t^2]\right\} dx \\
 &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}[(x-t)^2 - t^2]\right\} dx \\
 &= \exp\left\{\frac{t^2}{2}\right\} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}(x-t)^2\right\} dx}_{= 1}
 \end{aligned}$$

$$\frac{d}{dt} \exp\left\{\frac{t^2}{2}\right\} \Big|_{t=0} = t \exp\left\{\frac{t^2}{2}\right\} \Big|_{t=0} = 0$$

$$\frac{d}{dt^{2k+1}} \exp\left\{\frac{t^2}{2}\right\} \Big|_{t=0} = 0 \quad (\text{check})$$

In general, check: $X \sim N(\mu, \sigma^2)$

$$M_X(t) = E[e^{tx}] = \exp\left\{t\mu + \frac{1}{2}t^2\sigma^2\right\}$$

Result Example: $X \sim \text{Gamma}(\alpha, \beta)$

$$f(x|\alpha, \beta) = \frac{1}{\Gamma(\alpha)} \frac{1}{\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, \quad x > 0$$

$$\begin{aligned} E[e^{tx}] &= M_x(t) = \int_0^\infty e^{tx} \frac{1}{\Gamma(\alpha)} \frac{1}{\beta^\alpha} x^{\alpha-1} e^{-x/\beta} dx \\ &= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty x^{\alpha-1} \exp\left\{tx - x/\beta\right\} dx \\ &= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty x^{\alpha-1} \exp\left\{-x\left(\frac{1}{\beta} - t\right)\right\} dx \\ &= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty x^{\alpha-1} \exp\left\{-\frac{x}{\left[\frac{1}{\beta} - t\right]}\right\} dx \end{aligned}$$

This integral is finite if $\frac{1}{\beta} > t$

Thus the moment generating function does not exist for all $t \in (-\infty, \infty)$
It only exists when $t < \frac{1}{\beta}$.

when $t < \frac{1}{\beta}$

$$\begin{aligned} &= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty \frac{\left(\frac{1}{\beta} - t\right)^\alpha}{\left(\frac{1}{\beta} - t\right)^\alpha} x^{\alpha-1} \exp\left\{-\frac{x}{\left[\frac{1}{\beta} - t\right]}\right\} dx \\ &= \frac{1}{\beta^\alpha \left(\frac{1}{\beta} - t\right)^\alpha} \int_0^\infty \frac{1}{\Gamma(\alpha)} \left(\frac{1}{\beta} - t\right)^\alpha x^{\alpha-1} \exp\left\{-\frac{x}{\left[\frac{1}{\beta} - t\right]}\right\} dx \end{aligned}$$

Since it is the integral of
the density Gamma($\alpha, \frac{1}{\beta} - t$)

$$= \frac{1}{(1 - \beta t)^\alpha}$$

So, we have seen two cases.

In case 1, MGF exists for all t .

In case 2, MGF exists only for $t < \frac{1}{\beta}$.

Change of variable theorem:

$x_1, \dots, x_n \stackrel{iid}{\sim} f(x|\theta)$

the joint density of x_1, \dots, x_n is

$$f(x_1, \dots, x_n | \theta) = f(x_1 | \theta) \cdots f(x_n | \theta)$$

Question: Find the joint density of

$$u_1 = \psi_1(x_1, \dots, x_n), \dots, u_n = \psi_n(x_1, \dots, x_n)$$

Find the joint density of (u_1, \dots, u_n) from

the joint density of x_1, \dots, x_n .

$n=2, x_1, x_2 \stackrel{iid}{\sim} f(x|\theta)$

$$u_1 = x_1 + x_2, u_2 = 3(x_1 + x_2)$$

Recap:

① $x_1, \dots, x_n \stackrel{iid}{\sim} f(x|\theta)$, x_i 's are mutually independent and each $x_i \sim f(x|\theta)$

② $M_X(t) = E[e^{tX}]$

change of variable theorem

$x_1, \dots, x_n \stackrel{iid}{\sim} f(x|\theta)$

Goal: Find the joint density of u_1, \dots, u_n , where

$u_1 = \psi_1(x_1, \dots, x_n), \dots, u_n = \psi_n(x_1, \dots, x_n)$.

The joint density exists when

$(x_1, \dots, x_n) \rightarrow (u_1, \dots, u_n)$ is one-to-one

If $f_u(u_1, \dots, u_n)$ denotes the joint density of

u_1, \dots, u_n , then

$$f_u(u_1, \dots, u_n) = \left[\prod_{i=1}^n f(H_i(u_1, \dots, u_n) | \theta) \right] \left| \det \left(\left(\frac{\partial H_i(u_1, \dots, u_n)}{\partial u_j} \right) \right)_{i,j=1}^n \right|,$$

here, $X_i = H_i(u_1, \dots, u_n)$

Example: $u_1, u_2 \stackrel{iid}{\sim} U(0,1)$

$$X_1 = \sqrt{-2 \log(u_1)} \cos(2\pi u_2)$$

$$X_2 = \sqrt{-2 \log(u_1)} \sin(2\pi u_2)$$

Goal: Joint density of x_1 and x_2 .

$$\Psi_1(u_1, u_2) = \sqrt{-2 \log u_1} \cos(2\pi u_2)$$

$$\Psi_2(u_1, u_2) = \sqrt{-2 \log u_1} \sin(2\pi u_2)$$

$$u_i = H_i(x_1, x_2)$$

~~$$x_1 = \sqrt{-2 \log u_1} \cos(2\pi u_2)$$~~

~~$$x_2 = \sqrt{-2 \log u_1} \sin(2\pi u_2)$$~~

$$\tan(2\pi u_2) = \frac{x_2}{x_1} \Rightarrow u_2 = \frac{1}{2\pi} \tan^{-1}\left(\frac{x_2}{x_1}\right)$$

$$\text{also, } x_1^v + x_2^v = -2 \log u_1 \Rightarrow u_1 = \exp\left\{-\frac{1}{2}(x_1^v + x_2^v)\right\}$$

$$u_1 = H_1(x_1, x_2) = \exp\left\{-\frac{1}{2}(x_1^v + x_2^v)\right\}$$

$$u_2 = H_2(x_1, x_2) = \frac{1}{2\pi} \tan^{-1}\left(\frac{x_2}{x_1}\right)$$

$$J = \begin{pmatrix} \frac{\partial H_1(x_1, x_2)}{\partial x_1} & \frac{\partial H_1(x_1, x_2)}{\partial x_2} \\ \frac{\partial H_2(x_1, x_2)}{\partial x_1} & \frac{\partial H_2(x_1, x_2)}{\partial x_2} \end{pmatrix}$$

$$= \begin{pmatrix} -x_1 \exp\left\{-\frac{(x_1^v + x_2^v)}{2}\right\} & -x_2 \exp\left\{-\frac{(x_1^v + x_2^v)}{2}\right\} \\ \frac{1}{2\pi} \frac{1}{\left(1 + \frac{x_2^v}{x_1^v}\right)} \left(-\frac{x_2}{x_1^v}\right) & \frac{1}{2\pi} \frac{1}{\left(1 + \frac{x_2^v}{x_1^v}\right)} \left(\frac{1}{x_1}\right) \end{pmatrix}$$

$$\det(J) = -\frac{1}{2\pi} \frac{1}{\left(1 + \frac{x_2^v}{x_1^v}\right)} \exp\left\{-\frac{(x_1^v + x_2^v)}{2}\right\}$$

$$-\frac{1}{2\pi} \frac{1}{\left(1 + \frac{x_2^v}{x_1^v}\right)} \left(\frac{x_2^v}{x_1^v}\right) \exp\left\{-\frac{(x_1^v + x_2^v)}{2}\right\}$$

$$|\det(J)| = \frac{1}{2\pi} \left[\frac{1}{1 + \frac{x_2^{\sim}}{x_1^{\sim}}} + \frac{1}{\left(1 + \frac{x_2^{\sim}}{x_1^{\sim}}\right)} \left(\frac{x_2^{\sim}}{x_1^{\sim}}\right) \right] \exp\left\{-\frac{(x_1^{\sim} + x_2^{\sim})}{2}\right\}$$

$$= \frac{1}{2\pi} \exp\left\{-\frac{(x_1^{\sim} + x_2^{\sim})}{2}\right\}$$

$$f(u_1, u_2) = 1, \quad 0 < u_1 < 1, \quad 0 < u_2 < 1$$

$$f(x_1, x_2) = \frac{1}{2\pi} \exp\left\{-\frac{(x_1^{\sim} + x_2^{\sim})}{2}\right\}, \quad 0 < \exp\left\{-\frac{1}{2}(x_1^{\sim} + x_2^{\sim})\right\} < 1,$$

$$0 < \frac{1}{2\pi} \tan^{-1}\left(\frac{x_2}{x_1}\right) < 1$$

$$= \frac{1}{2\pi} \exp\left\{-\frac{(x_1^{\sim} + x_2^{\sim})}{2}\right\}, \quad -\infty < x_1 < \infty,$$

$$-\infty < x_2 < \infty$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x_1^{\sim}}{2}\right\} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x_2^{\sim}}{2}\right\}$$

$\Rightarrow x_1 \sim N(0, 1)$, $x_2 \sim N(0, 1)$ and x_1 and x_2 are independent.

$$u_1, u_2 \stackrel{iid}{\sim} U(0, 1) \Rightarrow x_1, x_2 \stackrel{iid}{\sim} N(0, 1)$$

BOX-MULLER TRANSFORMATION.

Some Important Results on Random Sample

① x_1, \dots, x_n is a random sample ~~with~~ having the moment generating function (MGF) as $M(t)$. What is the MGF of $\frac{1}{n} \sum_{i=1}^n x_i = \bar{x}$?

$$\textcircled{2} M_{\bar{x}}(t) = E[e^{t\bar{x}}] = E[e^{\frac{1}{n} t \sum_{i=1}^n x_i}]$$

$$= E[e^{\frac{t}{n} x_1}] \cdots \cdots E[e^{\frac{t}{n} x_n}]$$

$$= M\left(\frac{t}{n}\right) \cdots \cdots M\left(\frac{t}{n}\right) = \left[M\left(\frac{t}{n}\right)\right]^n$$

② x and y are independent random variables with p.d.f. $f_x(x)$ and $f_y(y)$ respectively. Then the p.d.f. of $Z = x+y$ is

$$f_Z(z) = \int f_x(w) f_y(z-w) dw$$

$$\text{Pf: } P(Z \leq z) = P(x+y \leq z) = \int_{-\infty}^z P(w+y \leq z) f_x(w) dw$$

$$P(x+y \leq z) = \int P(x+y \leq z | x=w) f_x(w) dw$$

$$= \int P(w+y \leq z | x=w) f_x(w) dw$$

$$= \int P(w+y \leq z) f_x(w) dw$$

$$= \int P(y \leq z-w) f_x(w) dw$$

(4)

$$= \int \left\{ \int_{-\infty}^{z-w} f_y(y) dy \right\} f_x(w) dw$$

$y = y' + w \Rightarrow$ when $y = z - w$ then $y' = z$

$y = y' - w \Rightarrow$ when $y = z - w$ then $y' = z$

$$= \int \left\{ \int_{-\infty}^z f(y' - w) dy' \right\} f_x(w) dw$$

$$= \int_{-\infty}^z \left\{ \int f_y(y' - w) f_x(w) dw \right\} dy'$$

$$\Rightarrow P(\textcircled{x} z \leq z) = \int_{-\infty}^z \left\{ \int f_y(y' - w) f_x(w) dw \right\} dy'$$

$$f_Z(z) = \int f_y(z-w) f_x(w) dw$$

$$\textcircled{3} \quad x_1, \dots, x_n \stackrel{iid}{\sim} N(0, 1), \quad \sum_{i=1}^n x_i \sim \chi_n \sim \text{Gamma}\left(\frac{n}{2}, 2\right)$$

$$\textcircled{4} \quad x_1, \dots, x_n \stackrel{iid}{\sim} N(0, \sigma^2), \quad \frac{\sum_{i=1}^n x_i}{\sigma} \sim \bar{X}_n$$

$$\textcircled{5} \quad x_1, \dots, x_n \stackrel{iid}{\sim} N(\mu, \sigma^2), \quad \text{let } \bar{X} = \frac{1}{n} \sum_{i=1}^n x_i$$

$$S^2 = \frac{1}{(n-1)} \sum_{i=1}^n (x_i - \bar{X})^2. \quad \text{Then}$$

(a) \bar{X} and S^2 are independent.

$$(b) \bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

$$(c) (n-1) \frac{S^2}{\sigma^2} \sim \chi_{n-1}^2$$

(5)

Pf:- Define an orthogonal matrix A s.t.

$$A = \begin{pmatrix} \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

An orthogonal matrix satisfies $AA' = A'A = I$.

A' = transpose of A.

Orthogonal matrix satisfies

- (1) Inner product between any two rows is 0.
- (2) Inner product ~~between~~ of a row with itself is 1.

$$(a_{k1}, \dots, a_{kn}) \cdot \left(\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}}\right) = 0$$

$$\Rightarrow \frac{1}{\sqrt{n}} \sum_{i=1}^n a_{ki} = 0 \quad \forall k = 2, \dots, n.$$

and if $k \neq l$ then, $\sum_{i=1}^n a_{ki} a_{li} = 0$

Now, lets define $\underline{z} = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = A \underline{x}$

$$\textcircled{*} \quad \underline{z} = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}, \quad \underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$\Rightarrow \underline{z}' \underline{z} = \underline{x}' A' A \underline{x} = \underline{x}' \underline{x}$$

$$\Rightarrow \sum_{i=1}^n z_i^2 = \sum_{i=1}^n x_i^2 \quad \text{--- --- --- --- --- (*)}$$

(6)

$$z_1 = \left(\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}} \right) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$= \frac{1}{\sqrt{n}} \sum_{i=1}^n x_i = \frac{1}{\sqrt{n}} n \bar{x} = \sqrt{n} \bar{x}$$

Hence from (*)

$$\begin{aligned} \sum_{i=1}^n z_i^2 &= \sum_{i=1}^n x_i^2 \Rightarrow z_2^2 + \dots + z_n^2 = \sum_{i=1}^n x_i^2 - z_1^2 \\ &= \sum_{i=1}^n x_i^2 - n \bar{x}^2 \\ &= \sum_{i=1}^n (x_i - \bar{x})^2 \\ &= (n-1) s^2 \end{aligned}$$

We have shown that \bar{x} is only a function of z_1 and s^2 is only a function of z_2, \dots, z_n .

Goal: To show z_1, z_2, \dots, z_n are independent.

$$z_R = \sum_{i=1}^n a_{Ri} x_i, \quad z_L = \sum_{i=1}^n a_{Li} x_i$$

$$\text{cov}(z_R, z_L) = \text{cov}\left(\sum_{i=1}^n a_{Ri} x_i, \sum_{i=1}^n a_{Li} x_i\right)$$

$$= \sum_{i=1}^n a_{Ri} a_{Li} \text{var}(x_i) = \tau^2 \sum_{i=1}^n a_{Ri} a_{Li} = 0$$

Here z_R and z_L are both normally distributed as they are linear combinations of normal random variables x_i 's.

Thus $\text{cov}(z_R, z_L) = 0 \Rightarrow z_R, z_L$ are independent.

Similarly z_1 can be shown to be independent of the other z_i 's.

$\Rightarrow \bar{x}$ and s^2 are independent.

(b) ~~$\bar{x} \sim N(\mu, \frac{\sigma^2}{n})$~~
 $x_1, \dots, x_n \stackrel{iid}{\sim} N(\mu, \sigma^2), \quad \bar{x} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$

(c) $(n-1)s^2 = z_2^2 + \dots + z_n^2$

$$z_R = \sum_{i=1}^n a_{ki} x_i$$

$$E[z_R] = E\left[\sum_{i=1}^n a_{ki} x_i\right] = \sum_{i=1}^n a_{ki} E[x_i] - \mu \sum_{i=1}^n a_{ki} = 0$$

$$\text{Var}(z_R) = \text{Var}\left(\sum_{i=1}^n a_{ki} x_i\right) = \sum_{i=1}^n a_{ki}^2 \sigma^2 = \sigma^2$$

$$\Rightarrow z_2, \dots, z_n \stackrel{iid}{\sim} N(0, \sigma^2)$$

$$\Rightarrow (n-1)\frac{s^2}{\sigma^2} = \frac{z_2^2 + \dots + z_n^2}{\sigma^2} \sim \chi_{n-1}^2$$

Recap: Some distributions and results related to those distributions.

① $x_1, \dots, x_n \stackrel{iid}{\sim} N(0, \sigma^2)$, $\frac{\sum_{i=1}^n x_i^2}{\sigma^2} \sim \chi^2_n$

② $x_1, \dots, x_n \stackrel{iid}{\sim} N(\mu, \sigma^2)$ and $\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$,
 $s^2 = \frac{1}{(n-1)} \sum_{i=1}^n (x_i - \bar{x})^2$ then

(a) \bar{x} and s^2 are indep.

→ (b) $\bar{x} \sim N(\mu, \frac{\sigma^2}{n})$ (c) $\frac{s^2(n-1)}{\sigma^2} \sim \chi^2_{n-1}$.

Some important distributions:

$x_1, \dots, x_n \stackrel{iid}{\sim} N(\mu, \sigma^2)$ and σ^2 is known

$\frac{\sqrt{n}(\bar{x} - \mu)}{\sigma} \sim N(0, 1)$ by result (b).

data for rainfall in 10 days in SC are x_1, \dots, x_{10} . Suppose $x_1, \dots, x_n \stackrel{iid}{\sim} N(\mu, \sigma^2)$ and we have been told that $\sigma^2 = 45 \text{ mm}^2$.

• $\mu = 200 \text{ mm}$.

you use z-test to test if $\mu = 200 \text{ mm}$.

$Z = \frac{\sqrt{10}(\bar{x} - 200)}{\sqrt{45}}$ and we check if $|Z| > 1.96$

• When σ^2 is unknown, we can't use the z statistic to test. we will use

$$T = \frac{\sqrt{n}(\bar{x} - \mu)}{S} = \frac{\sqrt{n}(\bar{x} - \mu)/\sigma}{\sqrt{\frac{(n-1)s^2}{(n-1)\sigma^2}}} = \frac{U}{\sqrt{\frac{V}{n-1}}}$$

where $U = \sqrt{n}(\bar{x} - \mu)/\sigma$ and $V = \frac{(n-1)s^2}{\sigma^2}$

by the previous result, $U \sim N(0,1)$, $V \sim \chi_{n-1}^2$
and U, V are independent.

one has to find the distribution of

$$T = \frac{U}{\sqrt{\frac{V}{n-1}}}$$

the distribution can be found by the change of variable theorem and this dist. is known as the Student's t-distribution. The density of a Student's t-dist. with p degrees of freedom, denoted by t_p , is given by

$$f(t) = \frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2})\sqrt{p\pi}} \left(1 + \frac{t^2}{p}\right)^{-\frac{(p+1)}{2}}, \quad -\infty < t < \infty.$$

For $p=1$, no moment exists for t_p , but for $p>1$, $E[t_p] = 0$ and $\text{Var}(t_p) = \frac{p}{p-2}$, for $p>2$.

In our case $T = \frac{U}{\sqrt{\frac{V}{n-1}}} \sim t_{n-1}$

F-distribution:

$$X_1, \dots, X_n \stackrel{iid}{\sim} N(\mu_1, \sigma_1^2)$$

$$Y_1, \dots, Y_m \stackrel{iid}{\sim} N(\mu_2, \sigma_2^2)$$

~~the~~ the goal is to test whether $\sigma_1^2 = \sigma_2^2$

$$U = \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{\sigma_1^2} \sim \chi_{n-1}^2, \quad V = \frac{\sum_{j=1}^m (Y_j - \bar{Y})^2}{\sigma_2^2} \sim \chi_{m-1}^2$$

$$F = \frac{u/(n-1)}{v/(n-1)}$$

under the assumption that

$$\tilde{\sigma}_1^2 = \tilde{\sigma}_2^2 \text{ this quantity } F = \frac{\tilde{x}_{n-1}^2}{\frac{\tilde{x}_{m-1}^2}{m-1}}$$

and the numerator and denominator are independent.

Def: If $u \sim \chi_p^2$ and $v \sim \chi_q^2$ and u, v are independent then, $\frac{u/p}{v/q}$ follows an F-distribution denoted by $F_{p,q}$

$$(a) x \sim F_{p,q} \Rightarrow \frac{1}{x} \sim F_{q,p}$$

$$(b) x \sim t_q \Rightarrow x^2 \sim F_{1,q}$$

$$(c) \text{ If } x \sim F_{p,q} \Rightarrow \frac{\left(\frac{p}{q}\right)x}{1 + \left(\frac{p}{q}\right)x} \sim \text{Beta}\left(\frac{p}{2}, \frac{q}{2}\right)$$

Order Statistics:

$x_1, \dots, x_n \stackrel{iid}{\sim} f(x|\theta)$ then the order statistics of the random sample are given by

$$x_{(1)} = \min_{1 \leq i \leq n} x_i, \quad x_{(2)}, \dots, x_{(n)} = \max_{1 \leq i \leq n} x_i$$

thus, $x_{(1)} \leq x_{(2)} \leq \dots \leq x_{(n)}$ are the order statistics from the random sample

Joint distribution of all the order statistics

$$f_{X_{(1)}, \dots, X_{(n)}}(x_1, \dots, x_n | \theta) = n! f(x_1 | \theta) \cdots f(x_n | \theta), \quad x_1 < x_2 < \dots < x_n.$$

Marginal density of $X_{(j)}$, i.e. the j th order statistic

$$f_{X_{(j)}}(x) = n c_1 \cdot \cancel{n-1} c_{j-1} f(x | \theta) [F(x | \theta)]^{j-1} [1 - F(x | \theta)]^{n-j}$$

$$(F(x | \theta) = \text{CDF of } f = \int_{-\infty}^x f(y | \theta) dy)$$

$$= \frac{n (n-1)!}{(j-1)! (n-j)!} f(x | \theta) [F(x | \theta)]^{j-1} [1 - F(x | \theta)]^{n-j}$$

$$= \frac{n!}{(j-1)! (n-j)!} f(x | \theta) [F(x | \theta)]^{j-1} [1 - F(x | \theta)]^{n-j}$$

Example: $x_1, \dots, x_n \stackrel{iid}{\sim} \text{Exp}(\lambda)$

x_i : Lifetime of the i -th bulb.

You would like to know the distribution of that bulb which becomes ineffective first.

• $X_{(1)}$, so need to find the distribution of $X_{(1)}$.

$$f_{X_{(1)}}(x) = \frac{n!}{(n-1)! \lambda} \exp(-x/\lambda) \left[1 - \exp\left(-\frac{x}{\lambda}\right)\right]^0 \left[\exp\left(-\frac{x}{\lambda}\right)\right]^{n-1}, \quad x > 0.$$

The density of $x_{(j)}$ is

$$f_{X_{(j)}}(x) = \frac{n!}{(j-1)! (n-j)!} \frac{1}{\lambda} \exp(-\frac{x}{\lambda}) \left[1 - \exp(-\frac{x}{\lambda})\right]^{j-1} \left[\exp(-\frac{x}{\lambda})\right]^{n-j}, \quad x > 0.$$

Joint density of $X_{(i)}$ and $X_{(j)}$, $i < j$

$$\begin{aligned} f_{X_{(i)}, X_{(j)}}(x_1, x_2) &= n c_2 x_2^{n-2} c_{i-1}^{(n-2-i+1)} c_{j-i-1}^{n-j} \\ &\quad f(x_1|\theta) f(x_2|\theta) \left[F(x_1|\theta)\right]^{i-1} \left[1 - F(x_2|\theta)\right]^{n-j} \\ &\quad \left[F(x_2|\theta) - F(x_1|\theta)\right]^{j-i-1}, \quad x_1 \leq x_2 \\ &= \frac{n!}{(i-1)! (j-i-1)! (n-j)!} f(x_1|\theta) f(x_2|\theta) \left[F(x_1|\theta)\right]^{i-1} \\ &\quad \left[1 - F(x_2|\theta)\right]^{n-j} \left[F(x_2|\theta) - F(x_1|\theta)\right]^{j-i-1}, \quad x_1 \leq x_2. \end{aligned}$$

Example: X_1, \dots, X_n iid $\text{Exp}(\lambda)$.

$$\begin{aligned} f_{X_{(i)}, X_{(j)}}(x_1, x_2) &= \frac{n!}{(i-1)! (j-i-1)! (n-j)!} \left[\frac{1}{\lambda^2} \exp(-x_1/\lambda) \exp(-x_2/\lambda) \right] \\ &\quad \times \left[1 - \exp(-x_1/\lambda)\right]^{i-1} \left[\exp(-x_2/\lambda)\right]^{n-j} \\ &\quad \left[\exp(-x_1/\lambda) - \exp(-x_2/\lambda)\right]^{j-i-1}, \quad x_1 \leq x_2. \end{aligned}$$

Convergence of random variables

Let there be a sequence of random variables x_1, \dots, x_n, \dots . How to define convergence of this sequence of random variables?

Notions of convergence: We will describe two different notions for the convergence of this sequence.

Convergence in Probability:

A sequence of random variables x_1, \dots, x_n, \dots converges in probability to a random variable x , if for every $\epsilon > 0$, $\lim_{n \rightarrow \infty} P(|x_n - x| > \epsilon) = 0$.

(*) Some natural results that come are following.

(*) Note that, we mathematically write it as

$$x_n \xrightarrow{P} x$$

(a) (*) If $x_n \xrightarrow{P} x$, $y_n \xrightarrow{P} y$, $x_n + y_n \xrightarrow{P} x + y$.

$$P(|x_n + y_n - x - y| > \epsilon) \leq P(|x_n - x| > \epsilon/2) + P(|y_n - y| > \epsilon/2)$$

$$\{ |x_n + y_n - x - y| > \epsilon \} \subseteq \{ |x_n - x| > \epsilon/2 \} \cup \{ |y_n - y| > \epsilon/2 \}$$

$$\lim_{n \rightarrow \infty} P(|x_n + y_n - x - y| > \epsilon) \leq \lim_{n \rightarrow \infty} P(|x_n - x| > \epsilon/2) + \lim_{n \rightarrow \infty} P(|y_n - y| > \epsilon/2) \\ = 0$$

$$\Rightarrow \lim_{n \rightarrow \infty} P(|x_n + y_n - x - y| > \epsilon) = 0$$

$$\Rightarrow x_n + y_n \xrightarrow{P} x + y .$$

(b) If $x_n \xrightarrow{P} x$ and g is a continuous function
then $g(x_n) \xrightarrow{P} g(x)$